HYBRID PRIKRY FORCING

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ABSTRACT. We present a new forcing notion combining diagonal super-compact Prikry focing with interleaved extender based forcing. We start with a supercompact cardinal κ. In the final model the cofinality of κ is ω, the singular cardinal hypothesis fails at κ and GCH holds below κ. Moreover we define a scale at κ, which has a stationary set of bad points in the ground model.

1. INTRODUCTION

Groundbreaking works of Cohen and Easton showed that every reasonable behavior of the powerset operation for regular cardinals is consistent. In contrast, for singular cardinals, there are deep ZFC constraints on the powerset function, and consistency results require large cardinals. This leads to a long standing project in set theory, known as *the Singular Cardinal Problem:* find a complete set of rules for the behavior of the operation $\kappa \mapsto 2^\kappa$ for singular cardinals κ.

Obtaining consistency results about singular cardinals involves violating *the singular cardinal hypothesis* (SCH). SCH states that if κ is singular strong limit, then $2^\kappa = \kappa^+$. One classical method of constructing a model where SCH fails is to blow up the power set of a large cardinal, and then singularize it. Then κ remains strong limit, but GCH does not hold below κ. The reason for that is that by reflection, adding many subsets of κ in advance requires adding many subsets of α for a measure one set of α's below κ.

So this construction does not achieve what we can refer to as "the ultimate failure" of SCH: having a singular cardinal κ, such that $2^\kappa > \kappa^+$ and $\text{GCH}_{<\kappa}$ holds. The same is true for Magidor's original supercompact Prikry forcing, with which he first showed that SCH at $\aleph_\omega$ can be violated. Starting with a cardinal κ that is λ-supercompact, supercompact Prikry forcing singularizes all cardinals in the interval $[\kappa, \lambda]$. An important variation of this is diagonal supercompact Prikry forcing, which singularizes cardinals in the interval $[\kappa, \lambda)$, where λ is a successor of a singular cardinal.

Another approach is to start with a cardinal that is already singular and a limit of strong cardinals and then add many Prikry sequences via extender based forcing to increase its power set. Extender based forcing is one of the most direct ways to violate SCH, and it starts with strong cardinals in the

ground model. It first appeared in Gitik-Magidor [4]. Since no subsets are added in advance, GCH below κ can be maintained.

Here we describe a a construction that combines both strategies. More precisely, we define a *hybrid Prikry forcing* that simultaneously singularizes a large cardinal κ, singularizes and collapses an infinite interval of cardinals above κ, and uses extenders to add many Prikry sequences to $\prod_n \kappa$, so that SCH is violated. This way, since we are not adding subsets in advance, and our main forcing does not add bounded subsets of κ, we can maintain GCH below κ. Our forcing combines diagonal supercompact Prikry forcing with extender based forcing. The former is used to singularize κ, adding a generic sequence $\langle x_n \mid n < \omega \rangle$, where each $x_n \in \mathcal{P}_\kappa(\kappa^{+n})$. The latter is used to add many Prikry sequences though $\prod_n \kappa \cap x_{n+1}$.

Theorem 1.1. *Suppose that κ is supercompact. Then there is a forcing notion, $\mathbb{P}$, which we call the hybrid Prikry, such that:*

(1) *$\mathbb{P}$ does not add bounded subsets of κ,*
(2) *setting $\mu := (\kappa^{+\omega+1})^V$, we have that $\mathbb{P}$ preserves cardinals $\tau \geq \mu$,*
(3) *$\mathbb{P}$ adds an ω sequence cofinal in κ and makes μ the successor of κ,*
(4) *$\mathbb{P}$ adds μ^+ many new ω - sequences in $\prod_n \kappa$.*

Finer analysis shows:

Theorem 1.2. *Suppose in V, κ is supercompact and GCH holds. Let $\mu = \kappa^{+\omega+1}$. Then after forcing with the hybrid Prikry, in the generic extension we have:*

(1) *κ is singular of cofinality ω, μ is the successor of κ, and cardinals above μ are preserved.*
(2) *GCH holds below κ, and $2^\kappa = \kappa^{++}$. And so SCH fails at κ.*

Moreover, there is a scale at κ, whose set of bad points is stationary in the ground model.

Scales are a central concept in PCF theory. Given a singular cardinal $\kappa = \sup_n \kappa_n$, where each κ_n is regular, a *scale of length κ^+* is a sequence of functions $\langle f_\alpha \mid \alpha < \kappa^+ \rangle$ in $\prod_n \kappa_n$ that is increasing and cofinal with respect to the eventual domination ordering, $<^*$. I.e. $f <^* g$ if for all large n, $f(n) < g(n)$. A point $\alpha < \kappa^+$ with $\mathrm{cf}(\alpha) > \omega$ is *good* if there is an unbounded $A \subset \alpha$ such that $\{f_\beta(n) \mid \beta \in A\}$ is strictly increasing for all large n. Otherwise α is a *bad point*. A scale is *good* if on a club every point of uncountable cofinality is good, and a scale is *bad* if it is not good, i.e. there are stationary many bad points. The existence of a bad scale is a reflection type property. For example, every scale above a supercompact cardinal is bad.

The paper is organized as follows. In section 2 we define the main forcing and prove some of its main properties, including the Prikry property, cardinal preservation, and violating SCH. In section 3 we define the scale.

2. THE FORCING

Suppose that in V, GCH holds and κ is a supercompact cardinal and set $\mu := \kappa^{+\omega+1}$. Let U be a normal measure on $\mathcal{P}_\kappa(\mu)$, and for all $n < \omega$, let U_n be the projection of U to $\mathcal{P}_\kappa(\kappa^{+n})$. Also let $\sigma : V \to M$ witness that κ is $\kappa^{+\omega+2} + 1$ - strong and let $E = \langle E_\alpha \mid \alpha < \kappa^{+\omega+2} \rangle$ be κ complete ultrafilters on κ, where $E_\alpha = \{Z \subset \kappa \mid \alpha \in \sigma(Z)\}$. As in [2] we define a strengthening of the Rudin-Keisler order: for $\alpha, \beta < \kappa^{+\omega+2}$, set $\alpha \leq_E \beta$ if $\alpha \leq \beta$ and there is a function $f : \kappa \to \kappa$, such that $\sigma(f)(\beta) = \alpha$. For $\alpha \leq_E \beta$, fix projections $\pi_{\beta\alpha} : \kappa \to \kappa$ to witness this ordering, setting $\pi_{\alpha,\alpha}$ to be the identity. We do this as in Section 2 of [2] with respect to κ, so that we have:

(1) $\sigma\pi_{\beta\alpha}(\beta) = \alpha$.
(2) For all $a \subset \kappa^{+\omega+2}$ with $|a| < \kappa$, there are unboundedly many $\beta < \kappa^{+\omega+2}$, such that $\alpha <_E \beta$ for all $\alpha \in a$.
(3) For $\alpha < \beta \leq \gamma$, if $\alpha \leq_E \gamma$ and $\beta \leq_E \gamma$, then $\{\nu < \kappa \mid \pi_{\gamma,\alpha}(\nu) < \pi_{\gamma,\beta}(\nu)\} \in E_\gamma$.
(4) If $\{\alpha_i \mid i < \tau\} \subset \alpha < \kappa^{+\omega+2}$ with $\tau < \kappa$, are such that for all $i < \tau$, $\alpha_i <_E \alpha$, then there is $A \in E_\alpha$, such that for all $\nu \in A$, for all $i, j < \tau$, if $\alpha_i \leq_E \alpha_j$, then $\pi_{\alpha,\alpha_i}(\nu) = \pi_{\alpha_j,\alpha_i}(\pi_{\alpha,\alpha_j}(\nu))$.

Definition 2.1. *The poset $\mathbb{Q} = \mathbb{Q}_0 \cup \mathbb{Q}_1$ is defined as follows:*
$\mathbb{Q}_1 = \{f : \kappa^{+\omega+2} \rightharpoonup \kappa \mid |f| < \kappa^{+\omega+1}\}$ *and $\leq_1$ is the usual ordering. $\mathbb{Q}_0$ has conditions of the form $p = \langle a, A, f \rangle$ such that:*

- *$a \subset \kappa^{+\omega+2}$, $|a| < \kappa$, and $\beta \leq_E \max(a)$ for all $\beta \in a$,*
- *$f \in \mathbb{Q}_1$ and $a \cap \operatorname{dom}(f) = \emptyset$,*
- *$A \in E_{\max a}$*
- *for all $\alpha \leq_E \beta \leq_E \gamma$ in a, and $\nu \in \pi_{\max a,\gamma}"A$, $\pi_{\gamma,\alpha}(\nu) = \pi_{\beta,\alpha}(\pi_{\gamma,\beta}(\nu))$.*
- *for all $\alpha < \beta$ in a, for all $\nu \in A$, $\pi_{\max a,\alpha}(\nu) < \pi_{\max a,\beta}(\nu)$*

$\langle b, B, g \rangle \leq_0 \langle a, A, f \rangle$ if:

(1) *$b \supset a$,*
(2) *$\pi_{\max b \max a}"B \subset A$,*
(3) *$g \supset f$.*

Define $\leq^* = \leq_0 \cup \leq_1$ and for $p, q \in \mathbb{Q}$, $p \leq q$, if $p \leq^* q$ or $p \in \mathbb{Q}_1$, $q = \langle a, A, f \rangle \in \mathbb{Q}_0$ and:

(1) *$p \supset f$, $a \subset \operatorname{dom}(p)$*
(2) *$p(\max a) \in A$*
(3) *for all $\beta \in a$, $p(\beta) = \pi_{\max a,\beta}(p(\max a))$*

Basically $\mathbb{Q}$ is the Prikry type forcing notion $\mathbb{Q}_n$ from Section 2 of [2] with κ replacing κ_n. Note that $\mathbb{Q}_1$ is dense in $\mathbb{Q}$, and $\mathbb{Q}_1$ is equivalent to the Cohen poset for adding $\kappa^{+\omega+2}$ many subsets to $\kappa^{+\omega+1}$. In particular, we have the following:

Proposition 2.2. $\mathbb{Q}$ *has the $\kappa^{+\omega+2}$ chain condition.*

We also remark that just forcing with $\mathbb{Q}_0$ will collapse $\kappa^{+\omega+1}$ to κ (see Assaf Sharon's thesis [9]).

Definition 2.3. *For a condition $p = \langle a, A, f \rangle \in \mathbb{Q}_0$ and $\nu \in A$, let $p^\frown \nu = f \cup \{\langle \beta, \pi_{\max a, \beta}(\nu) \rangle \mid \beta \in a\}$. I.e. $p^\frown \nu$ is the weakest extension of p in $\mathbb{Q}_1$ with ν in its range.*

Note that if $g \in \mathbb{Q}_1$, with $g \leq p = \langle a, A, f \rangle$, there is a unique $\nu \in A$ such that $g \leq p^\frown \nu$ (take $\nu = g(\max a)$). Furthermore, if $g \leq q \leq p$, $q = \langle a^q, A^q, f^q \rangle$, $p = \langle a^p, A^p, f^p \rangle$, and $g \leq q^\frown \nu$, then $g \leq p^\frown \nu'$, where $\nu' = \pi_{\max a^q, \max a^p}(\nu)$.

Proposition 2.4. $\mathbb{Q}$ *has the Prikry property. I.e. given a condition p and a formula in the forcing language ϕ, there is $q \leq^* p$ such that q decides ϕ.*

Proof. The proof is standard and appears in [2]. We include it for completeness. Let $p = \langle a, A, f \rangle$ be a condition and ϕ be a formula. For each $\nu \in A$, let $g_\nu \leq p^\frown \nu$ be such that $g_\nu \parallel \phi$ and set $f_\nu = g_\nu \restriction (\mathrm{dom}\, g_\nu \setminus a)$.

Since the domain of the f_ν's is bigger than the size of A, we can arrange that the f_ν's are compatible. Shrink A to a set $A' \in E_{\max a}$ such that for all $\nu \in A'$, g_ν decides ϕ the same way. Let $f' = \bigcup_{\nu \in A'} f_\nu$. Then $p' = \langle a, A', f' \rangle$ decides ϕ.

$\square$

We are ready to define the main forcing. For $x, y \in \mathcal{P}_\kappa(\kappa^{+\omega})$, we will denote $\kappa_x = \kappa \cap x$ and use the notation $x \prec y$ to mean $x \subset y$ and $o.t.(x) < \kappa_y$. Since on a measure one set, κ_x is an inaccessible cardinal, we assume this is always the case.

Definition 2.5. *Conditions in $\mathbb{P}$ are of the form*
$$p = \langle x_0, f_0, ..., x_{l-1}, f_{l-1}, A_l, F_l, A_{l+1}, F_{l+1}... \rangle$$
where $l = length(p)$ and:
> (1) *For $n < l$,*
>> (a) *$x_n \in \mathcal{P}_\kappa(\kappa^{+n})$, and for $i < n$, $x_i \prec x_n$,*
>> (b) *$f_n \in \mathbb{Q}_1$.*
> (2) *For $n \geq l$,*
>> (a) *$A_n \in U_n$, and $x_{l-1} \prec y$ for all $y \in A_l$.*
>> (b) *F_n is a function with domain A_n, for $y \in A_n$, $F_n(y) \in \mathbb{Q}_0$.*
> (3) *For $x \in A_n$, denote $F_n(x) = \langle a_x^n, A_x^n, f_x^n \rangle$. Then for $l \leq n < m$, $y \in A_n, z \in A_m$ with $y \prec z$, we have $a_y^n \subset a_z^m$.*

For a condition p, we will use the notation $p = \langle x_0^p, f_0^p, ..., x_{l-1}^p, f_{l-1}^p, A_l^p, F_l^p, ... \rangle$, and for $n \geq l$ and $y \in A_n^p$, $F_n^p(y) = \langle a(F_n^p(y)), A(F_n^p(y)), f(F_n^p(y)) \rangle$. The stem of p is $h = \langle x_0^p, f_0^p, ..., x_{l-1}^p, f_{l-1}^p \rangle$.

For two conditions p, q, set $q \leq p$ if $p = \langle x_0, f_0^p, ..., x_{n-1}, f_{n-1}^p, A_n^p, F_n^p, ... \rangle$, $q = \langle x_0, f_0^q, ..., x_{n+m-1}, f_{n+m-1}^q, A_{n+m}^q, F_{n+m}^q, ... \rangle$, and:
> (1) *For $i < n$, $f_i^q \supset f_i^p$.*
> (2) *For $i < m$, $x_{n+i} \in A_{n+i}^p$.*

(3) *For $i < m$, $f_{n+i}^q \leq_{\mathbb{Q}} F_{n+i}^p(x_{n+i})$ and if $\nu < \kappa$ is the unique such that $f_{n+i}^q \leq_{\mathbb{Q}} F_{n+i}^p(x_{n+i})^\frown \nu$, then:*
  - *if $i < m - 1$, we have $\nu < \kappa_{x_{n+i+1}}$,*
  - *if $i = m - 1$, we have $\nu < \kappa_z$ for all $z \in A_{n+m}^q$.*

(4) *For $i \geq n + m$, $A_i^q \subset A_i^p$ and for all $y \in A_i^q$, $F_i^q(y) \leq_{\mathbb{Q}} F_i^p(y)$.*

We say that q is a direct extension of p, denoted by $q \leq^* p$, if $q \leq p$ and $lh(q) = lh(p)$.

Sometimes we will say that we *shrink* a condition p to mean replacing p with a direct extension.

Lemma 2.6. $\langle \mathbb{P}, \leq^* \rangle$ *is κ-closed.*

Proof. Let $\tau < \kappa$ and $\langle p_\alpha \mid \alpha < \tau \rangle$ be a $\leq^*$-decreasing sequence in $\mathbb{P}$ of conditions with some fixed length l. Let $\vec{x} = \langle x_0, ..., x_{l-1} \rangle$ be such that for some (equivalently all) α, $\text{stem}(p_\alpha) = \langle \vec{x}, \vec{f}^{p_\alpha} \rangle$. First let f_i be stronger than each $f_i^{p_\alpha}$ for $i < l$. Also, for $n \geq l$, let $A_n = \bigcap_{\alpha < \tau} A_n^{p_\alpha}$

Next we will define $\langle F_n \mid l \leq n < \omega \rangle$ by induction on n, such that each F_n has domain A_n, for $x \in A_n$, $F_n(x) = \langle a(F_n(x)), A(F_n(x)), f(F_n(x)) \rangle \in \mathbb{Q}_1$. We will maintain that for all $l \leq k < n$, $x \in A_k, y \in A_n$ if $x \prec y$, then $a(F_k(x)) \subset a(F_n(y))$. Note that this implies that $a(F_k(x)) \cap \text{dom}(f(F_n(y))) = \emptyset$.

Let $d = \bigcup_{\alpha < \tau, l \leq n < \omega, x \in A_n} \text{dom}(f(F_n^{p_\alpha}(x)))$. Then d is a bounded subset of $\kappa^{+\omega+2}$. Note that taking lower bounds of elements in $\mathbb{Q}_0$ requires more than just taking the union of the first coordinate i.e. the $a's$. We also have to take a maximal element. Thus when defining the lower bound, we will make sure that the maximal element of each $a(F_k(x))$ is above $\max(d)$. To do that we use that there are always unboundedly many choices for a maximal element.

Fix n and suppose we have defined F_k for all $l \leq k < n$. For $y \in A_n$, let $a'_y = \bigcup_{\alpha < \tau} a(F_n^{p_\alpha}(y))$, and $a''_y = \bigcup_{k < n, x \in A_k, x \prec y} a(F_k(x))$. Let $\rho > \max(d)$ be a maximal element for $a'_y \cup a''_y$ and set $a_y = a'_y \cup a''_y \cup \{\rho\}$.

Finally set $f_y = \bigcup_{\alpha < \tau} f(F_n^{p_\alpha}(y))$ and A_y to be the intersection of all $\pi^{-1}_{\rho, \max(a(F_n^{p_\alpha}(y)))} A(F_n^{p_\alpha}(y))$ for $\alpha < \tau$. Then define $F_n(y) = \langle a_y, A_y, f_y \rangle$.

Claim 2.7. $a_y \cap \text{dom}(f_y) = \emptyset$

Proof. By construction $\rho \notin \text{dom}(f_y)$. Now, suppose that $\xi \in a(F_n^{p_\alpha}(y))$ for some $\alpha < \tau$. Then for any $\beta < \tau$, setting $\gamma = \max(\alpha, \beta)$, we have $a(F_n^{p_\alpha}(y)) \subset a(F_n^{p_\gamma}(y))$ and $\text{dom}(f(F_n^{p_\beta}(y))) \subset \text{dom}(f(F_n^{p_\gamma}(y)))$. Since $a(F_n^{p_\gamma}(y)) \cap \text{dom}(f(F_n^{p_\gamma}(y))) = \emptyset$, we have $\xi \notin \text{dom}(f((F_n^{p_\gamma}(y)))$. It follows that $a'_y \cap \text{dom}(f_y) = \emptyset$.

Finally, to show that $a''_y \cap \text{dom}(f_y) = \emptyset$, we argue that for all $k < n$, $x \in A_k$ with $x \prec y$, $a(F_k(x)) \cap \text{dom}(f_y) = \emptyset$. Use induction on k. Denote $a(F_k(x)) = a'_x \cup a''_x \cup \{\max(a(F_k(x)))\}$, where a'_x, a''_x are defined as above but for x. By construction the maximal element is not in the domain of f_y.

Also since $x \prec y$, $a'_x \subset a'_y$, and so it is disjoint from $\mathrm{dom}(f_y)$. Lastly, since $z \prec x \prec y$ implies $z \prec y$, by induction we have that $a(F_m(z)) \cap \mathrm{dom}(f_y) = \emptyset$ for any $m < k, z \in A_m, z \prec x$. So we have that $a''_x \cap \mathrm{dom}(f_y) = \emptyset$.

This concludes the argument that $a''_y \cap \mathrm{dom}(f_y) = \emptyset$, and finishes the claim. $\qquad\square$

Finally define p by setting $p = \langle x_0, f_0, ..., x_{l-1}, f_{l-1}, A_l, F_l, A_{l+1}, F_{l+1}, ...\rangle$. Then p is a lower bound.

$\qquad\square$

Next we show that $\mathbb{P}$ has the Prikry property. First we introduce some notation.

Definition 2.8. *Let p be a condition with length l. For $y \in A_l^p$ and $\nu \in A(F_l^p(y))$, define $p^\frown\langle y, \nu\rangle =:$*

$$\langle x_0^p, f_0^p, ..., x_{l-1}^p, f_{l-1}^p, y, F_l^p(y)^\frown\nu, A_{l+1}^{y,\nu}, F_{l+1}^{y,\nu}, ...\rangle$$

where for $n > l$:
- $A_n^{y,\nu} = A_n^p \cap \{z \mid y \prec z, \nu < \kappa_z\}$,
- $F_n^{y,\nu} = F_n^p \restriction A_n^{y,\nu}$.

Similarly, for any $n > l$, $\vec{y} = \langle y_l \prec ... \prec y_n\rangle$ of points in $\prod_{l \le i \le n} A_i^p$, and $\vec{\nu} \in \prod_{l \le i \le n} A(F_i^p(y_i))$ with each $\nu_i < \kappa_{y_{i+1}}$, define $p^\frown\langle \vec{y}, \vec{\nu}\rangle$ to be the weakest extension of p with length $n + 1$ such that the stem is derived from $\vec{y}$ and $\vec{\nu}$.

Also, if p, q are conditions, $n < \mathrm{lh}(p), \mathrm{lh}(q)$, we say that $[F_n^p]_{U_n} \le [F_n^q]_{U_n}$ if for almost all $x \in A_n^p$, $F_n^p(x) \le_{\mathbb{Q}_0} F_n^q(x)$.

Lemma 2.9. *(Diagonal lemma) Suppose that p is a condition with length l, and for all $y \in A_l^p$ and $\nu \in A(F_l^p(y))$, there are conditions $p^{y,\nu} \le^* p^\frown\langle y, \nu\rangle$, such that:*

(1) *For all $i < l$, $y_1, y_2 \in A_l^p$ and $\nu_1 \in A(F_l^p(y_1)), \nu_2 \in A(F_l^p(y_2))$, we have that*
    $f_i^{p^{y_1,\nu_1}}$ *and* $f_i^{p^{y_2,\nu_2}}$ *are compatible.*
(2) *for each $y \in A_l^p$, $\langle f_l^{y,\nu} \restriction (\mathrm{dom}(f_l^{y,\nu}) \setminus a_l^p(y)) \mid \nu \in A(F_l^p(y))\rangle$ are pairwise compatible.*
(3) *For all n, $\langle [F_n^{p^{y,\nu}}]_{U_n} \mid y \in A_l^p, \nu \in A(F_l^p(y))\rangle$ are pairwise compatible.*

Then there is $p' \le^* p$ such that if $q \le p'$ with $\mathrm{lh}(q) \ge \mathrm{lh}(p) + 1$, then $q \le p^{y,\nu}$ for some y, ν.

Proof. Denote $p = \langle x_0, f_0, ..., x_{l-1}, f_{l-1}, A_l, F_l, ...\rangle$, $F_n(y) = \langle a_y^n, A_y^n, f_y^n\rangle$ and each $p^{y,\nu} = \langle x_0, f_0^{y,\nu}, ..., x_{l-1}, f_{l-1}^{y,\nu}, y, f_l^{y,\nu}, A_{l+1}^{y,\nu}, F_{l+1}^{y,\nu}, ...\rangle$. Also denote $F_n^{y,\nu}(z) = \langle a_n^{y,\nu}(z), A_n^{y,\nu}(z), f_n^{y,\nu}(z)\rangle$.

Define $p' = \langle x_0, f_0', ..., x_{l-1}, f_{l-1}', A_l', F_l', ...\rangle$ as follows:

- $f_i' = \bigcup_{y \in A_l, \nu \in A_y^l} f_i^{y,\nu}$ for $i < l$.
- $A_l' = A_l$,
- $F_l'(y) = \langle a_y^l, A_y^l, \bigcup_{\nu \in A_y^l} f_l^{y,\nu} \restriction (\mathrm{dom}(f_l^{y,\nu}) \setminus a_y^l)\rangle$

- For $n > l$, $A'_n = \triangle_{y \in A_l, \nu \in A^l_y} A^{y,\nu}_n = \{z \in A_n \mid z \in \bigcap_{y \prec z, \nu \in A^l_y \cap \kappa_z} A^{y,\nu}_n\}$,
- For $n > l$, let $[F'_n]_{U_n}$ be stronger than each $[F^{y,\nu}_n]_{U_n}$ for $y \in A_l, \nu \in A^l_y$. Here we use that the number of such pairs is κ^{+l}, and $j_n(\mathbb{Q}_0)$ is closed under sequences of length κ^{+n}. By further shrinking A'_n we can arrange that for all $y \in A'_n$, $F'_n(y) \leq_{\mathbb{Q}} F^{z,\nu}_n(y)$ for all $z \prec y$ and $\nu \in A^l_z \cap \kappa_y$. Also, arguing as in Lemma 2.6 we arrange that the F'_n's satisfy the last item of the definition of $\mathbb{P}$.

Then p' is as desired.

$\square$

Corollary 2.10. *Let $0 < n < \omega$. For every condition p and every formula in the forcing language ϕ, there is $p' \leq^* p$, such that for all $q \leq p'$ with $lh(q) = n + lh(p)$, if there is $r \leq^* q$ which decides ϕ, then q decides ϕ.*

Proof. By induction on n. If $n = 0$, the result is immediate. So, suppose that $n > 0$, and the corollary holds for $n - 1$. Fix p and ϕ. For all $y \in A^p_{lh(p)}$ and $\nu \in A^p_{lh(p)}(y)$, by the inductive assumption there is $p^{y,\nu} \leq^* p^\frown \langle y, \nu \rangle$, such that for all $q \leq p^{y,\nu}$ with $lh(q) = n + lh(p)$, if there is $r \leq^* q$ which decides ϕ, then q decides ϕ.

Defining these condition inductively, we can arrange that they satisfy the assumptions of the diagonal lemma. Apply the diagonal lemma to the conditions $p^{y,\nu}$ and p to get $p' \leq^* p$, such that if $q \leq p'$ with $lh(q) \geq lh(p)+1$, then $q \leq p^{y,\nu}$ for some y, ν. Then p' is as desired. For if $q \leq p'$ with $lh(q) = n + lh(p)$, let y, ν be such that $q \leq p^{y,\nu}$. Now, if $r \leq^* q$ decides ϕ, then by the way we chose $p^{y,\nu}$, it follows that q decides ϕ.

$\square$

Lemma 2.11. *(The Prikry property) Suppose p is a condition and ϕ is a formula in the forcing language, then there is $q \leq^* p$ which decides ϕ.*

Proof. We start by showing two claims. The first claim states that we can restrict ourselves to a fixed length when looking at extensions of p deciding ϕ. The second claim applies the diagonal lemma to shrink p so that the weakest extensions of the fixed length decide ϕ.

Claim 2.12. *There is $lh(p) \leq n < \omega$ and $p' \leq^* p$, such that for all $q \leq^* p'$, there is $r \leq q$ with length n such that r decides ϕ.*

Proof. Suppose otherwise. I.e. we have that:

(†) for all $n \geq lh(p)$, for all direct extensions p' of p, there is $q \leq^* p'$, such that for all $r \leq q$ with length n, $r \nVdash \phi$.

We will build a decreasing sequence of conditions $\langle p^n \mid l \leq n < \omega \rangle$, where $l = lh(p)$, such that each $p^n \leq^* p$, and for all $r \leq p^n$ of length n, we have that r does not decide ϕ. Set $p_l = p$. Suppose $n > l$ and we have defined p^{n-1}. Let p^n be given by applying (†) to n and p^{n-1}. Finally, let q be stronger than every p^n. It follows that no $r \leq q$ decides ϕ. Contradiction.

$\square$

By the above claim and Corollary 2.10 we can shrink p and fix n so that:

- for all direct extensions p' of p, there is $q \leq p'$ of length n that decides ϕ,
- for all $q \leq p$ with $lh(q) = n$, if there is $r \leq^* q$ which decides ϕ, then q decides ϕ.

Assume for simplicity that $lh(p) = 1$ and $n = 3$. The general case is similar. Denote $p = \langle x_0, f_0, A_1, F_1, ... \rangle$, and for $y \in A, n > 0$, $F_n(y) = \langle a_y^n, A_y^n, f_y^n \rangle$. For $x \in A_1, y \in A_2, x \prec y$ and $\nu \in A_x^1$ let $B_{x,\nu,y}^+ = \{\delta \in A_y^2 \mid p^\frown \langle \langle x, y \rangle, \langle \nu, \delta \rangle \rangle \Vdash \phi\}$, $B_{x,\nu,y}^- = \{\delta \in A_y^2 \mid p^\frown \langle \langle x, y \rangle, \langle \nu, \delta \rangle \rangle \Vdash \neg\phi\}$, and $B_{x,\nu,y}^c = A_y^2 \setminus (B_{x,\nu,y}^+ \cup B_{x,\nu,y}^-)$. One of these sets is measure one; let $B_{x,\nu,y}$ be that measure one set.

Set $B_{x,y} = \bigcap_{\nu \in A_x^1 \cap \kappa_y} B_{x,\nu,y}$

Let $A_{x,\nu}^+ = \{y \in A_2 \mid B_{x,\nu,y} = B_{x,\nu,y}^+\}$, $A_{x,\nu}^- = \{y \in A_2 \mid B_{x,\nu,y} = B_{x,\nu,y}^-\}$, $A_{x,\nu}^c = A_2 \setminus (A_{x,\nu}^+ \cup A_{x,\nu}^+)$. One of these is measure one; let $A_{x,\nu}$ be that measure one set.

$A_2' = \triangle A_{x,\nu} = \{y \in A_2 \mid y \in \bigcap_{x \prec y, \nu < \kappa_y} A_{x,\nu}\}$. Define F_2' by $\text{dom}(F_2') = A_2'$ and each $F_1'(y) = \langle a_y, \bigcap_{x \prec y} B_{x,y}, f_y \rangle$.

Let $B_x^+ = \{\nu \in A_x^1 \mid A_{x,\nu} = A_{x,\nu}^+\}$, $B_x^- = \{\nu \in A_x^1 \mid A_{x,\nu} = A_{x,\nu}^-\}$, and $B_x^c = A_x^1 \setminus (B_x^+ \cup B_x^-)$.

Set $B_x = B_x^+$ if it is measure one, B_x^- if it is measure one, and B_x^c otherwise.

Let $A^+ = \{x \in A_1 \mid B_x = B_x^+\}$, $A^- = \{x \in A_1 \mid B_x = B_x^-\}$, $A^c = A_1 \setminus (A^+ \cup A^-)$. One of these is measure one; let A_1' be that measure one set. Define F_1' by $\text{dom}(F_1') = A_1'$ and $F_1'(x) = \langle a_x, B_x, f_x \rangle$.

Set $p' = \langle x_0, f_0, A_1', F_1', A_2', F_2' \rangle^\frown p \restriction [3, \omega)$. We will show that p' is as desired.

By the way we choose p, we can fix condition $r \leq p'$ with length 3, such that $r \parallel \phi$. We have to show that p' decides ϕ.

Claim 2.13. $A_1' = A^+$ or $A_1' = A^-$.

Proof. Let x, y, ν, δ, be such that $r \leq^* p'^\frown \langle \langle x, y \rangle, \langle \nu, \delta \rangle \rangle$. Then since p was chosen to satisfy Corollary 2.10 for $n = 2$, we have that $p'^\frown \langle \langle x, y \rangle, \langle \nu, \delta \rangle \rangle$ decides ϕ. Now, suppose for contradiction that $A_1' = A^c$, then $x \in A^c$ and so $B_x = B_x^c$. Then since $\nu \in B_x = B_x^c$, we have that $A_{x,\nu} = A_{x,\nu}^c$. Since $y \in A_2'$, $x \prec y$, and $\nu < \kappa_y$, we have that $y \in A_{x,\nu} = A_{x,\nu}^c$. So, $B_{x,\nu,y} = B_{x,\nu,y}^c$. Then $\delta \in B_{x,y} \subset B_{x,\nu,y} = B_{x,\nu,y}^c$. So, $p'^\frown \langle \langle x, y \rangle, \langle \nu, \delta \rangle \rangle$ does not decide ϕ. Contradiction. □

Then p' decides ϕ.

□

Corollary 2.14. $\mathbb{P}$ *does not add bounded subsets of* κ.

It follows that all cardinals less than or equal to κ are preserved and GCH holds below κ. Next we show that μ is preserved. We use the following fact.

Proposition 2.15. *Suppose that D is a dense set and p is a condition with length l. Then there is some n and $q \leq^* p$, such that for all $\vec{y} \in \prod_{l \leq i < n} A_i^p$, $\vec{\nu} \in \prod_{l \leq i < n} A_{y_i}^p$, we have that $q ^\frown \langle \vec{y}, \vec{\nu} \rangle \in D$*

Proof. This is essentially the Prikry property, so we only outline the proof. First by shrinking measure one sets, we may assume that for some fixed n, for all $q \leq p$ of length $n + l$, there is some $r \leq^* q$ such that $r \in D$. Then diagonalize over $\vec{y} \in \prod_{l \leq i < n} A_i^p$, $\vec{\nu} \in \prod_{l \leq i < n} A_i^p(y_i)$ to get a condition $q \leq^* p$ such that for all $\vec{y}, \vec{\nu}$, $q ^\frown \langle \vec{y}, \vec{\nu} \rangle$ is in D. $\qquad \square$

Let G be $\mathbb{P}$ generic, and let $\langle x_n^* \mid n < \omega \rangle$, each $x_n^* \in \mathcal{P}_\kappa(\kappa^{+n})$, be the added generic sequence. Set $\lambda_n = x_n^* \cap \kappa$. Standard density arguments yield the following.

Proposition 2.16.   (1) *if $\langle A_n \mid n < \omega \rangle \in V$ is a sequence of sets such that every $A_n \in U_n$, then for all large n, $x_n^* \in A_n$.*
(2) *$\bigcup_n x_n^* = (\kappa^{+\omega})^V$,*
(3) *for each $n \geq 0$, the cofinality of $(\kappa^{+n})^V$ in $V[G]$ is ω.*

Proposition 2.17. *$\mu := (\kappa^{+\omega+1})^V$ remains a cardinal after forcing with $\mathbb{P}$.*

Proof. Suppose otherwise. Then in $V[G]$ the cofinality of μ is less than κ. Let n and $p \in G$ with $lh(p) > n$ be such that $p \Vdash$ "$\dot{f} : \tau \to \mu$ is unbounded and $\tau < \dot{\lambda}_n$". For all $\gamma < \tau$, let $D_\gamma = \{q \leq p \mid (\exists \eta)(q \Vdash \dot{f}(\gamma) = \eta)\}$. Then D_γ is dense below p. For each $\gamma < \tau$, let $p^\gamma \leq^* p$ and n_γ be given by Proposition 2.15. By defining $\langle p^\gamma \mid \gamma < \tau \rangle$ inductively, we arrange that $\langle p^\gamma \mid \gamma < \tau \rangle$ is a decreasing sequence. Let p' be such that $p' \leq^* p^\gamma$ for all γ.

Fix γ and $\langle \vec{x}, \vec{\nu} \rangle$ with length n_γ compatible with p^γ. Let $\alpha_\gamma^{\langle \vec{x}, \vec{\nu} \rangle}$ be such that

$$p^\gamma {}^\frown \langle \vec{x}, \vec{\nu} \rangle \Vdash \dot{f}(\gamma) = \alpha_\gamma^{\langle \vec{x}, \vec{\nu} \rangle}.$$

Let $\alpha_\gamma = \sup_{\langle \vec{x}, \vec{\nu} \rangle} \alpha_\gamma^{\langle \vec{x}, \vec{\nu} \rangle} < \mu$ and let $\alpha = \sup_{\gamma < \tau} \alpha_\gamma < \mu$. Then each $p^\gamma \Vdash \dot{f}(\gamma) \leq \alpha_\gamma$, and so $p' \Vdash (\forall \gamma)(\dot{f}(\gamma) \leq \alpha)$. Contradiction. $\qquad \square$

Our next goal is to show that μ^+ is preserved. Let

$$\mathbb{P}_0 := \{\langle x_0^p, ..., x_{n-1}^p, A_n^p, ... \rangle \mid p \in \mathbb{P}\},$$

with the induced ordering from $\mathbb{P}$. Since conditions with the same stem are compatible, $\mathbb{P}_0$ has the μ-chain condition. Characterization of genericity of $\mathbb{P}_0$ is given by condition (1) above, i.e. the condition is both necessary and sufficient for a generic sequence. This follows by adapting Mathias' arguments in [7] to diagonal supercompact Prikry. Then we have that G generates a generic filter for $\mathbb{P}_0$. Next we show that $\mathbb{P}/\mathbb{P}_0$ has the μ^+-chain condition.

Lemma 2.18. *Suppose that G_0 is $\mathbb{P}_0$-generic over V. Then $\mathbb{P}/G_0$ has the μ^+ chain condition.*

Proof. Say G_0 generates the generic sequence $\langle x_n^* \mid n < \omega \rangle$. Suppose that $\langle p_\alpha \mid \alpha < \mu^+ \rangle$ are conditions in $\mathbb{P}/G_0$. Then there is an unbounded $S \subset \mu^+$, $k < \omega$, such that:

(1) for all $\alpha \in S$, $\mathrm{lh}(p_\alpha) = k$,
(2) for all $n < k$, $\{\mathrm{dom}(f_n^{p_\alpha}) \mid \alpha \in S\}$ forms a Δ-system, with f^{p_α} and f^{p_β} having the same values on the kernel for $\alpha, \beta \in S$,
(3) for all $n \geq k$, $\{a(F_n^{p_\alpha}(x_n^*)) \cup \mathrm{dom}(f(F_n^{p_\alpha}(x_n^*))) \mid \alpha \in S\}$ forms a Δ-system, with $f(F_n^{p_\alpha}(x_n^*))$ and $f(F_n^{p_\beta}(x_n^*))$ having the same values on the kernel for $\alpha, \beta \in S$. Also, $a(F_n^{p_\alpha}(x_n^*)) \cap \mathrm{dom}(f(F_n^{p_\beta}(x_n^*))) = \emptyset$.

Then for any $\alpha, \beta \in S$, $n \geq k$, $F_n^{p_\alpha}(x_n^*)$ and $F_n^{p_\alpha}(x_n^*)$ are compatible in $\mathbb{Q}_0$. To find a lower bound, just pick a maximal element for $a(F_n^{p_\alpha}(x_n^*)) \cup a(F_n^{p_\beta}(x_n^*))$ that is above $\sup(\mathrm{dom}(f(F_n^{p_\alpha}(x_n^*))))$ and $\sup(\mathrm{dom}(f(F_n^{p_\beta}(x_n^*))))$.

Let $\alpha, \beta \in S$. Since for all $n \geq k$, $F_n^{p_\alpha}(x_n^*)$ and $F_n^{p_\alpha}(x_n^*)$ are compatible, by genericity of the x_n^*'s it follows that for all large n, $B_n = \{x \in A_n^{p_\alpha} \cap A_n^{p_\beta} \mid F_n^{p_\alpha}(x)$ and $F_n^{p_\alpha}(x)$ are compatible$\} \in U_n$. Say, for all $n \geq k'$, $B_n \in U_n$.

Define a condition p with length k' as follows. For $n < k$, let $f_n = f_n^{p_\alpha} \cup f_n^{p_\beta}$ and for $k \leq n < k'$, let $F_n(x_n^*) \in \mathbb{Q}_0$ be stronger than $F_n^{p_\alpha}(x_n^*)$ and $F_n^{p_\alpha}(x_n^*)$, and then let $f_n \leq_{\mathbb{Q}} F_n(x_n^*)^\frown \nu$ for some $\nu < \kappa_{x_{n+1}^*}$. Since the conditions are in $\mathbb{P}/G_0$, we can take such a ν. Set $\mathrm{stem}(p) = \langle x_0^*, f_0, ..., x_{k'-1}^*, f_{k'-1} \rangle$. Also for $n \geq k'$, set $A_n^p = B_n$, and for $x \in A_n^p$, let $F_n^p(x)$ be stronger than $F_n^{p_\alpha}(x)$ and $F_n^{p_\alpha}(x)$. Then p is stronger than p_α and p_β. $\qquad \square$

It follows that forcing with $\mathbb{P}$ preserves μ^+. Next we show that in the generic extension κ has $\mu^+ = (\kappa^{+\omega+2})^V$ many subsets. We have the added generic functions $f_n : (\kappa^{+\omega+2})^V \to \kappa$ for each n. Define $t_\alpha(n) = f_n(\alpha)$. Then each $t_\alpha \in \prod_n \kappa$. Let (in $V[G]$) $F_n = \bigcup_{p \in G, l(p) \leq n} a_n^p(x_n)$. Here $F_n^p(x) = \langle a_n^p(x), A_n^p(x), f_n^p(x) \rangle$. Set $F = \bigcup_n F_n$.

Proposition 2.19.   (1) *If $\alpha < \beta$ are both in F, then $t_\alpha <^* t_\beta$.*
(2) *F is unbounded in $(\kappa^{+\omega+2})^V$.*

Proof. For (1), suppose that $\alpha < \beta$ are both in F. Let $p, q \in G$ be such that for all large n, $\alpha \in a_n^p(x_n)$, $\beta \in a_n^q(x_n)$. Let $r \in G$ be a common extension of p, q. Then for some k, for all $n \geq k$, $\{\alpha, \beta\} \subset a_n^r(x_n)$. So, if $r' \leq r$ is in G and has length $n+1$ for $n \geq k$, by the last condition of the definition of $\mathbb{Q}_0$, we get that $f_n^{r'}(\alpha) < f_n^{r'}(\beta)$. So, for all large n, $t_\alpha(n) < t_\beta(n)$.

For (2), suppose that $\beta < (\kappa^{+\omega+2})^V$. We claim that $D = \{p \mid (\exists \gamma < (\kappa^{+\omega+2}) \setminus \beta)(\gamma \in \bigcap_{n \geq \mathrm{lh}(p), y \in A_n^p} a_n^p(y))\}$ is dense. For if p is a condition, let $\gamma \in (\kappa^{+\omega+2})^V \setminus (\bigcup_{n \geq \mathrm{lh}(p), y \in A^p} (a_n^p(y) \cup \mathrm{dom}(f_n^p(y))))$, such that $\beta < \gamma$. Then extend each $F_n^p(y)$ to obtain $\langle a_y^n, A_y^n, f_y^n \rangle$, such that $\gamma \in a_y^n$. Let $q \leq p$ be such that $F_n^q(y) = \langle a_y^n, A_y^n, f_y^n \rangle$. Then $q \in D$. Now, let $r \in D \cap G$, then r witnesses that there is $\gamma \in F$ such that $\gamma > \beta$. $\qquad \square$

Remark 1. Using the above and by the definition of the ordering of the forcing we can show that that if $\alpha \in F$, then for each n, $t_\alpha \in \prod_n \lambda_{n+1}$.

Remark 2. We can use $F \cap \mu$ to define a good scale as described in [1].

It follows that in $V[G]$, $2^\kappa = (\kappa^{+\omega+2})^V = (\kappa^{++})^{V[G]}$. So, SCH fails at κ.

3. The scale

Fix a scale $\langle g_\alpha^* \mid \gamma < \mu \rangle \in V$ in $\prod_n \kappa^{+n+1}$. Set $S := \{\gamma < \mu \mid \gamma$ is a bad point for $\langle g_\alpha^* \mid \gamma < \mu \rangle\}$. Since κ is supercompact in V, by standard reflection arguments S is stationary in V. In this section we define a scale at κ in the generic extension, such that every $\gamma \in S$ is bad for this new scale.

First we show a bounding lemma. We will use it to make sure that the scale we define in $V[G]$ is indeed cofinal. Recall that G generates the generic sequence $\langle x_n^* \mid n < \omega \rangle$ and we defined $\lambda_n = \kappa_{x_n^*}$ for $n < \omega$.

Lemma 3.1. *Suppose that in $V[G]$, $h \in \prod_n \lambda_n^{+n+1}$. Then there is a sequence of functions $\langle H_n \mid n < \omega \rangle$ in V, such that $\mathrm{dom}(H_n) = \mathcal{P}_\kappa(\kappa^{+n})$, $H_n(x) < \kappa_x^{+n+1}$ for all x, and for all large n, $h(n) < H_n(x_n^*)$.*

Proof. Let p force that $\dot{h}$ is as in the statement of the lemma. For simplicity assume that the length of p is 0.

Fix $n < \omega$ and $x \in A_n^p$. For all $\vec{z} \in \prod_{i \leq n} A_i^p$, $\vec{\nu} \in \prod_{i \leq n} A_i^p(z_i)$ of length $n + 1$ with $z_n = x$, $p^\frown \langle \vec{z}, \vec{\nu} \rangle \Vdash \dot{f}(n) < \kappa_x^{+n+1}$. Here z_n denotes the last element of $\vec{z}$. By the Prikry property we can build a decreasing sequence $\langle q^\gamma \mid \gamma < \kappa_x^{+n+1} \rangle$, such that for each γ, $q^\gamma \leq^* p^\frown \langle \vec{z}, \vec{\nu} \rangle$ and q^γ decides "$\dot{h}(n) = \gamma$". Let $q^{\langle \vec{z}, \vec{\nu} \rangle} \leq^* q^\gamma$, for each γ. Then $q^{\langle \vec{z}, \vec{\nu} \rangle}$ decides the value of $\dot{h}(n)$. By defining the $q^{\langle \vec{z}, \vec{\nu} \rangle}$'s inductively, we can arrange that they satisfy the assumptions of the diagonal lemma.

Define $H_n(x, \nu) = \sup\{\gamma < \kappa_x^{+n+1} \mid (\exists \vec{z}, \vec{\nu})(lh(\vec{z}) = \mathrm{lh}(\vec{\nu}) = n + 1, z_n = x, \nu_n = \nu, q^{\vec{z}, \vec{\nu}} \Vdash \dot{h}(n) = \gamma)\} + 1$. Here z_n and ν_n denote the last elements of $\vec{z}$ and $\vec{\nu}$ respectively.

Then $H_n(x, \nu) < \kappa_x^{+n+1}$. So, there is a measure one set $A_n(x)$, such that for all $\nu \in A_n(x)$ this has some constant value, denote it by $H_n(x)$. Let p' be obtained from p by shrinking the sets $A_n^p(x)$ to $A_n(x)$.

We apply the Diagonal Lemma to $q^{\langle \vec{z}, \vec{\nu} \rangle}$ for all $\langle \vec{z}, \vec{\nu} \rangle$ of length $n + 1$, and get $q^n \leq^* p'$ to be such that if $r \leq q^n$ has length at least $n + 1$, then for some $\langle \vec{z}, \vec{\nu} \rangle$, $r \leq q^{\langle \vec{z}, \vec{\nu} \rangle}$. Let q be stronger than each q^n. Then q forces that $\langle H_n \mid n < \omega \rangle$ is as desired.

□

The next lemma will be used to show that a witness of goodness in the generic extension gives rise to a witness of goodness in the ground model. In particular, if a point is bad in V, then it is bad in $V[G]$.

Lemma 3.2. *Let $\tau < \kappa$ be a regular uncountable cardinal in V (and so in $V[G]$), and suppose $V[G] \models A \subset ON, o.t.(A) = \tau$. Then there is a $B \in V$ such that B is an unbounded subset of A.*

Proof. Let $p \in G$, $p \Vdash \dot{h} : \tau \to \dot{A}$ enumerate $\dot{A}$. By the Prikry lemma, define a $\leq^*$-decreasing sequence $\langle p_\alpha \mid \alpha < \tau \rangle$, such for every $\alpha < \tau$, $p_\alpha \leq^* p$ and there is $n_\alpha < \omega$, such that every $q \leq p_\alpha$ with length n_α decides $\dot{h}(\alpha)$. Then there is an unbounded $I \subset \tau$ and $n < \omega$ such that for all $\alpha \in I$, $n = n_\alpha$. Let p' be stronger than all p_α for $\alpha < \tau$. By appealing to density, we may assume that $p' \in G$. Let $q \leq p$ be a condition in G with length n, and set $B = \{\gamma \mid (\exists \alpha \in I) q \Vdash \dot{h}(\alpha) = \gamma\}$. Then B is as desired. $\square$

Recall that $\mu = (\kappa^{+\omega+1})^V$ and that we fixed in advance a bad scale $\langle g_\beta^* \mid \beta < \mu \rangle$ in $\prod_n \kappa^{+n+1}$ in V, such that it has a stationary set of bad points, S of cofinality less than κ.

$\forall n < \omega$, $\forall \eta < \kappa^{+n+1}$, fix $f_n^\eta : X_n \longrightarrow V$, such that $\forall x f_n^\eta(x) < \kappa_x^{+n+1}$, and $[f_n^\eta]_{U_n} = \eta$. Define in $V[G]$, $\langle g_\beta \mid \beta < \mu \rangle$ in $\prod_n \lambda_n^{+n+1}$ by:

$$g_\beta(n) = f_n^{g_\beta^*(n)}(x_n^*)$$

Corollary 3.3. *$\langle g_\beta \mid \beta < \mu \rangle$ is a scale in $V[G]$, whose set of bad point is stationary in V.*

Proof. By the way we defined $\langle g_\beta \mid \beta < \mu \rangle$ and Lemma 3.1, we get that it is a scale (see for example the arguments in [1]). Also, if γ is a good point in $V[G]$ for $\langle g_\beta \mid \beta < \mu \rangle$ with cofinality τ with $\omega < \tau < \kappa$, then γ is a good point in V for $\langle g_\beta^* \mid \beta < \mu \rangle$. This follows from Lemma 3.2, which implies that if there is a witness for goodness in $V[G]$, then there is a witness for goodness in V.

$\square$

We conclude with some questions.

Question 1. *How much failure of square can we get in the final generic extension?*

In [5], it is shown that failure of weak square is consistent with not SCH at κ, but there GCH also fails below κ. It is open whether failure of SCH at κ together with GCH below κ is consistent with $\neg\square_\kappa^*$, or even with $\neg\square_{\kappa,\lambda}$ for all $\lambda < \kappa$. Another question concerns smaller cardinals:

Question 2. *Can we interleave collapses and obtain the present construction for $\kappa = \aleph_\omega$?*

A positive answer to the last question will probably involve using short extenders.

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